Social choice and information: the informational structure of uniqueness theorems in axiomatic social theories

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Abstract

The paper introduces the category of algebraic axioms and investigates when a social rule of decision-making can be uniquely characterized with such axioms. The first result shows that every set of axioms that characterize a given rule is equivalent to a set of three algebraic axioms. The second result suggests a method for constructing an algebraic proof of uniqueness via finding an appropriate path of maps. It says that we can characterize a rule if and only if we can find a path. Both theorems are then used to prove and analyze various characterization results in May’s binary social choice, Nash bargaining theory, and Sen’s social choice theory.

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1. Introduction

Axiomatic theories created with social interpretations in mind have flourished since Nash (1950), Arrow (1951), May (1952), and Shapley (1953) made their pioneering contributions. Today, the axiomatic method is applied in bargaining theory, in game theory (cooperative and noncooperative), in social choice theory (Arrow’s, Sen’s, May’s, the...
“Moscow school”, e.g., Aleskerov, 1997, and other versions), as well as in various theories of resource allocation or non-welfarist social choice. Many axioms of these theories are amazingly similar, as documented by various comparative classifications (e.g., Moulin, 1988: 1–5, Moulin and Thomson, 1997). Variants of anonymity, symmetry, monotonicity, comparability–measurability, or neutrality are prime examples of such universal axioms. This similarity is due to the fact that, within different frameworks, similar axioms deal with identical pieces of information, while partially ignoring information about the entire structure.

The goal of this paper is to find a convenient algebraic framework that would account for some similarities in axiomatic theories. The assumed model represents merely a skeleton of axiomatic social theories. It includes a domain of social problems and a solution space of social outcomes; a social rule assigns an outcome to every problem. Various specific theories can be obtained with a parametrization of problems with different individuals, alternatives, allocations, preference profiles, or utility functions. Similarities investigated in this paper are related to “uniqueness” or “characterization” theorems, i.e., results that say that there exists precisely one rule that satisfies a given set of axioms.

What makes the model specific enough is the interpretation of axioms. Three main categories of “algebraic” axioms—stationary, variance, and invariance—are introduced as the cornerstone of characterization exercises. A stationary axiom defines the value of a rule for one or more problems. A variance axiom says how the value of a rule changes when problem \( p \) is substituted with \( q \). An invariance axiom says that when \( p \) is substituted with \( q \), the solution does not change. These concepts were to a large extent motivated by Sen’s (1977a,b) program of comprehensive study of an informational basis of social choice theory and, in particular, by his idea of representing information with specific invariance assumptions. It turns out that invariance axioms are easily applicable to any axiomatic theory and can be naturally accompanied by the two remaining categories. Thus, the present paper provides a more general framework than Sen’s invariance-based social choice, while keeping the mathematics manageable enough to generate useful results.

Two theorems describe the properties of all characterization results. The first result says that every set of axioms that characterizes a rule uniquely can be substituted with an equivalent set of exactly three stationary–variance–invariance axioms. The second result provides a broader necessary and sufficient condition for the existence of a proof of uniqueness via a “path” of maps. This converts the task of axiomatic characterization into an algorithmic procedure. While both results are widely applicable, their consequences are most interesting for sets of axioms typically considered in social theories.

Practical usefulness of the results is conditional upon the degree of “algebraization” of the axioms of a given model. Thus, although all axioms can be “algebraized”, the benefits vary. Among the examples discussed in this paper, in the case of bargaining theory and standard Nash axioms, algebraization throws light on hidden regularities, and helps to construct intuitive proofs and to obtain a number of related results with virtually no effort. The characterization of the utilitarian rule by D’Aspremont and Gevers (1977) is amenable to analysis as well. On the other hand, in the case of May’s Theorem, the benefits are less impressive. For Shapley value, an algebraic proof is laborious and much longer than, for instance, Moulin’s (1995: 422–3) proof. This case is omitted in the paper. One benefit shared by all examples is the clarification of the structure of proof.
Section 2 introduces the primitives of our model, recalls basic notions of group theory, and defines formally the crucial concept of algebraic axiom. Section 3 discusses the algebraic content of familiar May’s axioms. Section 4 contains main results. Since the objects of study are specific uniqueness theorems from different models, a lot of attention is devoted to examples. Section 5 discusses how the present framework may be used to represent various results from May’s model of binary social choice, Nash bargaining model, and Sen’s model of social choice with various comparability–measurability assumptions. Section 6 concludes.

A word of caution closes the Introduction. Some confusion in social choice and related disciplines arises from the lack of distinction between a mathematical model and its empirical interpretation. I make no suggestion in this paper that axioms introduced here are compelling, natural, attractive, or mild, or that they represent rationality, equality, equity, or fairness in any ultimate way. The social choice interpretation given to the axioms and results is motivated solely by the fact that they are useful in analyzing and clarifying actual social axiomatic theories.

2. Model

Primitives:

\[ \Pi = \{ p_j \}_{j \in J} \] is a set of social problems, or problems, with \(|J| \geq 2\);
\[ O = \{ o_i \}_{i \in I} \] is a set of social outcomes, or outcomes, with \(|I| \geq 2\).

A model is a list of specific problems and outcomes \( M = (\Pi, O) \). Well-known examples of models include Arrowian social choice, May’s model of binary voting, Nash’s bargaining model, and Sen’s non-ordinal welfarist social choice. Individuals, alternatives, preferences, allocations, utility functions, etc., will be treated as parameters for \( \Pi \). A social rule, or simply a rule, is a function \( F: \Pi \rightarrow O \). The set of all rules is denoted by \( \Phi^M \). The following simple example is used for illustration.

**Example 1.** May’s binary social choice: \( \Pi^{M,n} = \{(d_1, \ldots, d_n): d_i \in \{-1,0,1\} \text{ for } i \in N = \{1, \ldots, n\}\}; \ O^{M} = \{-1,0,1\}. \)

Elements of \( \Pi^{M,n} \) are interpreted as lists of votes of \( n \) voters. Elements of \( O^{M} \) are interpreted as outcomes of voting. \( N \) is the set of voters. A voting rule \( F \) assigns to every list of votes an outcome of voting.

The notion of an axiom is central to this paper. In general, every axiom \( A \) considered below can be interpreted as a subset of rules, i.e., \( A \subseteq \Phi^M \). The intuition behind this interpretation is that any axiom \( A \) describing a property of a rule can be identified with the set of rules that satisfy \( A \). For simplicity, axioms whose purpose is to restrict the domain of rules are incorporated in the definition of rule. We will study an especially interesting subset of axioms that can be defined in purely algebraic form. Before we give definitions, basic concepts of group theory must be recalled.
For any nonempty set $X$, every one-to-one and onto function $\sigma: X \rightarrow X$ is called a \textit{permutation} of $X$. A family of permutations $G$ is a \textit{permutation group} for $X$, or a group, if (i) for any functions $\sigma_1, \sigma_2 \in G$, $G$ includes the \textit{composition} of functions $\sigma_1, \sigma_2$, (ii) $G$ includes an \textit{identity transformation} (a function $e$ such that for all $\sigma \in G$, $e \sigma = \sigma = e \sigma$), and (iii) for any function $\sigma \in G$, $G$ includes the \textit{inverse} of $\sigma$ (a function $\sigma^{-1}$ such that $\sigma^{-1} \sigma = \sigma \sigma^{-1} = e$). $G_1$ is a \textit{subgroup} of $G_2$ if $G_1 \subseteq G_2$ and $G_1$, $G_2$ are groups. We will denote such a relation by $G_1 \triangleleft G_2$. The group of all permutations is called the \textit{symmetric group} for $X$ and is denoted by $G^X$.

A \textit{partition} of $\Pi$, or a \textit{partition}, is any exhaustive family of pairwisely disjoint subsets of $\Pi$ and it is denoted by $\pi = \{P_t\}_{t \in T}$. Elements of $\pi$ are called orbits. The set of all problems that belong to the same orbit as $p_i$ is denoted by $\text{orb}(p_i)$. For any rule $F$, the partition $\text{Orb}(F)$ associated with $F$ is defined as follows: $\text{Orb}(F) = \{P \subset \Pi : P = F^{-1}(o)\}$ for some $o \in O$ and $P \neq \emptyset$ and is called the set of \textit{orbits} of $F$. Elements of $\text{Orb}(F)$ are maximal sets $P_t$ such that $F$ is constant over $P_t$. With every partition of $\Pi$, a permutation group is associated in the following way: $G(\pi) = \{\sigma \in G^X : \forall p_i \in \Pi, \sigma(p_i) \in \text{orb}(p_i)\}$. Every permutation of $G(\pi)$ operates only within the orbits of $\pi$. The group $G(\text{Orb}(F))$ is denoted by $G^F$. Finally, a set $\{P_t\}_{t \in T}$ is called a \textit{selection} from $\{P_t\}_{t \in T}$ if and only if $p_t^* \in P_t$ for all $t \in T$.

Now, we are ready to introduce the central concepts of this paper. A \textit{basic algebraic axiom} is any sentence of the following form: “For every rule $F \in \Phi^M$, $F(p_1) = o_1$, where $p_1 \in \Pi$, and $o_1 \in O$”, or “For every rule $F \in \Phi^M$, if $F(p_1) = o_1$ then $F(p_2) = o_2$, where $p_1, p_2 \in \Pi$, $p_1 \neq p_2$, and $o_1, o_2 \in O$”. A basic algebraic axiom either assigns some outcome to a problem or it says: “If $p_1$ is substituted by $p_2$, then $o_1$, the value of $F(p_1)$, is substituted by $o_2$”. An \textit{algebraic axiom} is any set of basic algebraic axioms. Such an axiom defines a specific “contagion” property.

Clearly, every algebraic axiom defines a subset of rules that satisfy the condition it imposes. When introducing a specific axiom, we will skip the quantifier “For every rule $F \in \Phi^M$” and the conditions related to $p_i$ and $o_i$.

The present definition of an axiom was chosen so to be sufficiently inclusive. Nevertheless, many axioms are more regular than is assumed in general and warrant special attention.

Our classification distinguishes among three categories of axioms. A \textit{stationary} axiom says that “$F(p_1) = o_1$”. A \textit{variance} axiom says that “If $F(p_1) = o_1$ then $F(p_2) = o_2 \neq o_1$”. An \textit{invariance} axiom says that “If $F(p_1) = o_1$ then $F(p_2) = o_1$”. It is clear from the definitions that every basic axiom is either a stationary axiom, a variance axiom, or an invariance axiom. Any algebraic axiom that consists exclusively of basic variance or invariance axioms is also called a variance or invariance axiom, respectively. Invariance axioms often state that the value of a social rule remains unchanged when a specific group of permutations is applied to the argument.

Some well-known axioms cannot be represented as algebraic axioms. Familiar examples include the Pareto principle and Arrow’s Independence of Irrelevant Alternatives. Both axioms substitute the single value $o_2$ that appears in the definition of basic axioms with a possibly large set $O_2$ which is a function of $p_1$, $o_1$, and $p_2$. Another important example is Shapley’s (1953) Additivity from his classical characterization of
what is known as Shapley value. The natural ‘contagion’ procedure here is of the form “If $F(p_1)=o_1$ and $F(p_2)=o_2$, then $F(p_3)=o_3$”, which is not an algebraic axiom.1

Finally, a set of axioms will be denoted by $\Omega$. To keep notation simple, we will refer to basic axioms that are members of axioms in $\Omega$ as if they were members of $\Omega$ itself.

3. Algebraic axioms: comment and examples

Under the most general, set-theoretic interpretation, every axiom can be viewed as a set of rules. If a set of such set-theoretic axioms $\Omega$ provides a unique axiomatic characterization of a rule, then the intersection of all its components must be a singleton. In the present framework, only a subset of all set-theoretic axioms is considered, and their content is interpreted in algebraic, not set-theoretic terms. The results of Section 4 establish that the class of algebraic axioms is equivalent to the class of all set-theoretic axioms in the sense that sets of set-theoretic axioms that characterize a rule uniquely are equivalent to some set of algebraic axioms.

Algebraic axioms intending to characterize a unique rule transmit information about the rule in a straightforward way. The smallest piece of information is provided by a single basic algebraic axiom. On the other hand, an axiom that provides full information specifies, for all $p$, the value of $F(p)$. Thus, such an axiom is equivalent to the definition of the rule.

It is helpful to consider the four axioms that were originally used by May (1952) in his celebrated characterization of simple majority rule. Below, numbers 1, $-1$, and 0 represent the vote for $x$, $y$, and indifference, respectively. If $p=(d_1, \ldots, d_n)$, then $\sigma_1(p)=\sigma_1(-d_1, \ldots, -d_n)$.

The first of May’s axioms, Decisiveness, is a domain-restriction axiom. It stipulates that the domain of the voting rule is unconstrained. In the present formulation, such axioms are implicitly assumed in the definition of rules. An explicit consideration of such axioms is possible, but it would lead to unnecessary complications. Setting Decisiveness aside, we can turn our attention to the remaining three axioms.

Axiom 1. Positive Responsiveness (PR): for all $p=(d_1, \ldots, d_n)\in \Pi$, if $F(d_1, \ldots, d_n)=0$ or 1, and $d_i'=d_i$ for all $i \neq j$, and $d_i'>d_j$, then $F(d_1', \ldots, d_n')=1$.

PR says that if $x$ ties or if $x$ is a winner, and one voter votes more favorably for $x$, then $x$ must be a winner under the new voting list.

PR can be divided into two subsets of basic axioms, with the variance part referring to $o_1=0$ and the invariance part referring to $o_1=1$.

Axiom 2. Neutrality (NT): for all $p=(d_1, \ldots, d_n)\in \Pi$, $F(p)=-F(-p)$.

NT includes a variance axiom and also a single basic stationary axiom that says “If $p=(0, \ldots, 0)$, then $F(p)=0$”.

An example of a variance axiom is Nash Scale Invariance. Various versions of symmetry provide examples of stationary axioms.

1I owe this observation to the referee.
Axiom 3. Anonymity (AN): for all $p=(d_1,\ldots,d_n)\in\Pi$, for all $\sigma\in G^N$, $F(p)=F(\sigma(p))$, where $\sigma(p)=(d_{\sigma(1)},\ldots,d_{\sigma(n)})$.

Anonymity is an invariance axiom that is connected to $G^{AN,II}$, the group that is isomorphic with the symmetric group of all permutations of the set of voters $N$, and which is defined as follows: $G^{AN,II}=${$e,\delta\in G^\Pi$: $\delta(d_1,\ldots,d_n)=(d_{\sigma(1)},\ldots,d_{\sigma(n)})$ for some $\sigma\in G^N$}. AN says that if a $\sigma\in G^{AN,II}$ is applied to $p$, the value of $F$ remains unchanged.

Well-known examples of invariance axioms connected to various groups of transformations are Sen’s measurability–comparability axioms. Every such axiom says that a social ordering selected by a Social Welfare Functional remains unchanged when the utility profile is transformed by a permutation from the subgroup of $G^\Pi$ that represents specific informational assumptions of measurability–comparability. Two of Sen’s axioms will be defined formally later in the context of Sen’s model of social choice. An example of an invariance axiom that is not based on a permutation group is provided by Nash’s Independence of Irrelevant Alternatives.

4. Results

In a typical characterization exercise, it is easy to check whether a given rule $F^*$ satisfies a set of axioms $\Omega$. The heart of a proof is to establish uniqueness. Thus, we will assume in the present section that a rule $F^*$ and a set of axioms $\Omega$ are given and that $F^*$ satisfies $\Omega$. Orb($F^*$) denotes the family of orbits of $F^*$ and orb($p$) is the orbit of $p$ in Orb($F^*$).

Theorem 1 states necessary and sufficient conditions for $\Omega$ to characterize $F^*$ uniquely in terms of algebraic axioms.

**Theorem 1.** Let $\{p^*_t\}_{t\in T}$ be a selection from Orb($F^*$) and $p^*_0\in\{p^*_t\}_{t\in T}$. Consider the following algebraic axioms:

- $A1.$ $F(p^*_0)=F^*(p^*_0)$;
- $A2.$ For every $t\in T$, if $F(p^*_0)=F^*(p^*_0)$ then $F(p^*_t)=F^*(p^*_t)$;
- $A3.$ For every $t\in T$, for all $p\in\Pi$ such that $p\in\text{orb}(p^*_t)$, $F(p)=F^*(p^*_t)$.

It holds that:

(i) $F^*$ uniquely satisfies $A1–A3$;
(ii) $\Omega$ uniquely characterizes $F^*$ if and only if $\Omega$ implies $A1–A3$; and
(iii) when $|\Pi|>|\text{Orb}(F^*)|$ and $|\text{Orb}(F^*)|\geq 2$, then $A1–A3$ are independent.

There is no attempt here to employ formal tools of modern mathematical logic to the extent greater than is customary in various social axiomatic theories. The only caveat is related to Gödel’s (1931) famous result that makes a distinction between true and provable propositions. For a consistent set of axioms, every provable proposition is true, but the opposite does not hold in general, i.e., in some circumstances there may exist a true proposition that cannot be proved. The statement “$\Omega$ characterizes $F^*$ uniquely” means in the context of this paper that we can prove that there exists a unique rule $F^*$ that satisfies $\Omega$ using standard techniques. Thus, all results of this paper deal with provable statements only.
Proof. (i) Since $F^*$ was assumed to satisfy A1–A3, we need to prove uniqueness. Take any $p \in \Pi$ and any $F$ that satisfies A1–A3. We have to show that $F(p) = F^*(p)$.

If $p = p_0^*$, then A1 implies $F(p) = F^*(p)$. Otherwise, take $p_j^* \notin \{ p_{1}^{*}, \ldots, p_{t}^{*} \}$ such that $p \in \text{orb}(p_j^*)$. If $p_j^* = p_0^*$, then $F(p) = F^*(p)$ is implied by A1 and A3. If $p_j^* \neq p_0^*$, then A1 and A2 imply that $F(p_j^*) = F^*(p_j^*)$, and A3 implies that $F(p) = F^*(p)$.

(ii) If $\Omega$ characterizes $F^*$, then $\Omega$ implies that $F = F^*$ and conditions A1–A3 are trivially satisfied. Thus, $\Omega$ implies A1–A3. If $\Omega$ implies A1–A3, then by (i) $\Omega$ implies that $F = F^*$.

(iii) For $i = 1, 2, 3$, we will construct a rule $F^i \neq F^*$ that satisfies all axioms from {A1, A2, A3} except for $A_i$.

Ad A1: Since $|\text{orb}(F)| \geq 2$, there is at least one permutation $\sigma^1 \in GO$ that defines a rule $F^1 = \sigma^1 F^*$ such that $F^1(p) = \sigma^1(F^*(p))$ and $F^1(p) \neq F^*(p)$ for all $p \in \Pi$. Thus, $F^1(p_0^*) \neq F^*(p_0^*)$.

Ad A2: since $|\text{orb}(F^*)| \geq 2$, there is at least one orbit $\text{orb}(p_1)$ that is different from $\text{orb}(p_0^*)$. Define $F^2$ as follows: $F^2(p_i) = F^*(p_0^*)$ for all $p_i \in \text{orb}(p_1)$; $F^2(p_i) = F^*(p_i)$ for all other $p_i$. By definition, $F^2(p_i) \neq F^*(p_i)$ for $p_i \in \text{orb}(p_1)$ and $F^2$ satisfies A1 since $F^2(p_0^*) = F^*(p_0^*)$. Thus, $F^2$ violates A2. $F^2$ satisfies A3 since it has exactly the same orbits as $F^*$ except for $\text{orb}(p_1)$ that was merged with $\text{orb}(p_0^*)$.

Ad A3: Since $|\Pi| > |\text{orb}(F^*)|$, there exist at least two different $p_2, p_3 \in \Pi$ such that $F^*(p_2) = F^*(p_3)$. Since $|\text{orb}(F^*)| \geq 2$, there exists an orbit of $F^*$, $\text{orb}(p_k)$, that is different from $\text{orb}(p_1)$. Consider first the case when $p_2 \notin \{ p_1^*, \ldots, p_t^* \}$. We can define $F^3$ as follows: $F^3(p_2) = F^*(p_0^*)$; $F^3(p_i) = F^*(p_i)$ for $p_i \neq p_2$. By construction, $F^3(p_2) \neq F^*(p_2)$ and the values assigned to $\{ p_i^* \} \in T$ are not affected. Thus, A1 and A2 are satisfied and A3 is violated. If $p_2 \in \{ p_i^* \} \in T$, then $p_1 \notin \{ p_1^* \} \in T$ and we can use $p_1$ to define $F^3$ in a similar way. □

Comment: first, (i) says that every rule $F^*$ is characterized uniquely by three algebraic axioms, a stationary, variance and invariance axiom. Any set of three axioms defined as in Theorem 1 will be called canonical. The meaning of canonical axioms is as follows:

A1 says that for some problem $p_0^*$, $F$ assumes the same value as $F^*$; A2 says that from the fact that $F$ and $F^*$ assume the same values for $p_0^*$ we can infer that $F$ and $F^*$ assume identical values for some element of every orbit of $F^*$; and A3 says that $F$ is constant within the orbits of $F^*$. In fact, as we will see in the next section, some axiomatic characterizations are based on slightly modified versions of axioms A1–A3. This result states a simple and general property of all functions, which turns out to be very useful in the context of social choice. Note that A1–A3 are constructed on the basis of $\text{orb}(F^*)$. This fact suggests a heuristic rule of thumb that the first step towards an axiomatic characterization of a rule should be calculating its orbits.

Second, (ii) says that for any set of axioms $\Omega$ that characterizes a rule $F^*$ uniquely, we can derive from $\Omega$ an equivalent set of canonical axioms that characterizes $F^*$ uniquely. As we will later see, such derivation is easy in some cases and quite laborious in other cases, depending on the structure of $\Omega$.

Finally, (iii) says that, except for trivial cases, axioms A1–A3 are independent.

Theorem 1 suggests the following proof of uniqueness of $F^*$. First, prove that $F(p_0^*) = F^*(p_0^*)$ for some $p_0^*$. Next, show that this fact implies that $F(p_i^*) = F^*(p_i^*)$ for all $p_i^*$ from some selection $\{ p_i^* \} \in T$ from $\text{orb}(F^*)$. Finally, show that the latter fact
implies $F(p_i) = F^*(p_i)$ for all $p_i \in \text{orb}(p^*_0)$. In other words, the task is to prove that the equality $F = F^*$ holds for some $p^*_0$, and that it is then “transmitted” to any $p_i$ through $p^*_0$. Before we formalize this intuition in the next result, we need a few new definitions.

Since $F^*$ satisfies $\Omega$, every stationary axiom in $\Omega$ must say that “$F(p) = F^*(p)$”, where $F^*(p)$ is a specific value from $O$. Other basic axioms say “If $F(p) = o_1$, then $F(q) = o_2$”. We will restrict our interest to basic axioms that begin with the clause “If $F(p) = F^*(p)$...”. Since $F^*$ satisfies $\Omega$, every such axiom must also end with the clause “... then $F(q) = F^*(q)$”. An axiom that says “$F(p) = F^*(p)$”, or “If $F(p) = F^*(p)$ then $F(q) = F^*(q)$” is called a map (given that $F^*$ and $\Omega$ are specified) and is denoted as $\omega_{pp}$ or $\omega_{pq}$, respectively.

A map says that $F$ and $F^*$ assume identical values for $p$ or that if $F(p)$ is equal to $F^*(p)$, then this equality is “transmitted” to $q$. Every set of basic maps is also called a map. Maps will be parametrized in a convenient way, depending on their domain and will be interpreted as functions. In the simplest case of a basic map, $\omega_{pq}$ is interpreted as a function $\omega_{pq}: \{q\} \rightarrow \Pi$ such that $\omega_{pq}(q) = p$. Note that the function goes in the opposite direction to the transmission of equality since we will be interested in finding, for a given $q$, some $p$ such that the equality $F = F^*$ is transmitted from $p$ to $q$. In many cases, the domain of a map will be equal to $\Pi$, and the map will be defined on the basis of some permutation from $G^\Pi$.

**Example 2.** Consider May’s PR axiom, $F^* =$ simple majority rule, $p = (0, \ldots, 0)$, and $q = (1, 0, \ldots, 0)$. Since $F$ may assume three values, $-1$, $0$, and $1$, there are nine possible basic axioms of the form “If $F(p) = x$ then $F(q) = y$”. Let us examine which of them are maps, given $F^*$ and PR.

Case 1. “If $F(p) = -1$ then $F(q) = 0$” is not a basic axiom included by PR since the precedents of basic axioms in PR are of the form “If $F(p) = 0$” or “If $F(p) = 1$” only.

Case 2. “If $F(p) = 0$ then $F(q) = 1$” is a map $\omega_{pq}$, i.e., it is a basic axiom in PR and $F(p) = F^*(p)$;

Case 3. “If $F(p) = 1$ then $F(q) = 1$” is a basic algebraic axiom in PR but is not a map since $F(p) \neq F^*(p)$.

It is easy to check the remaining six cases and see that “If $F(p) = 0$ then $F(q) = 1$” is a unique map among the nine possible basic axioms. In addition, “$F(q) = 1$” is the only map among the three basic axioms of the form “$F(q) = x$”.

**Definition 1.** For a given $\Omega$, $F^*$, and a nonempty set $\Pi_0 \subset \Pi$, a path of problems from $p$ to $\Pi_0$ consistent with $F^*$ and $\Omega$ is any sequence of problems $p_0, p_1, \ldots, p_n$ such that (a) $p_0 \in \Pi_0$ and $p_n = p$; (b) $\Omega$ implies that $F(q) = F^*(q)$ for all $q \in \Pi_0$; and (c) $\Omega$ implies that if $F(p_i) = F^*(p_i)$, then $F(p_{i+1}) = F^*(p_{i+1})$ for all $i = 0, \ldots, n - 1$.

When $p \in \Pi_0$, $p = p_0$ is a path. A path of maps from $p$ to $\Pi_0$ consistent with $F^*$ and $\Omega$ that is associated with a path of problems $p_0, p_1, \ldots, p_n$ is a sequence of maps $\omega_{p_0 p_1}, \ldots, \omega_{p_{n-1} p_n}$. We will call both sequences paths. A path of maps may contain terms of the form $\omega_{pp}$.
The next result states sufficient and necessary conditions for the unique characterization of $F^*$ by $\Omega$ in terms of finding an appropriate path. It is a modification of Theorem 1 into a more practical form.

**Theorem 2.** $\Omega$ uniquely characterizes $F^*$ if and only if for some nonempty set $\Pi_0 \subset \Pi$, for all $p \in \Pi$, there exists a path from $p$ to $\Pi_0$.

**Proof.** If: let $p \in \Pi$ be any problem, $p_0 \in \Pi_0$, and let $p_0, \ldots, p_n$ be a path from $p$ to $\Pi_0$. We have to show that $F(p) = F^*(p)$ for any $F$ that satisfies $\Omega$. From the definition of path, $F(p_0) = F^*(p_0)$. Thus, when $p = p_0$, we are done. Otherwise, since $F(p_i) = F^*(p_i)$ implies $F(p_{i+1}) = F^*(p_{i+1})$ for $i = 0, \ldots, n-1$, we infer by induction that $F(p_i) = F^*(p_i)$ for $i = 1, \ldots, n$. In particular, $F(p) = F(p_n) = F^*(p_n) = F^*(p)$.

Only if: let us take any $p \in \Pi$ and assume $\Pi_0 = \Pi$. Since $\Omega$ uniquely characterizes $F^*$, it must be for every $F$ that satisfies $\Omega$ that $F(p) = F^*(p)$. A path may be defined as $\omega_{pp}$.

Comment: Theorem 2 implies that when we can prove that $\Omega$ characterizes uniquely $F^*$, then we can also prove this fact by constructing a suitable $\Pi_0$ and finding a path for every $p \in \Pi$ to $\Pi_0$. It makes sense to call such a proof an algebraic proof. Theorem 1 suggests a particularly useful way of constructing such a proof. Set $\Pi_0$ may be a singleton \{ $p_0$ \}. A desired path may consist of $\omega_{p_0 p_0}$, then a segment of maps that link $p_0$ with a selection from all orbits of $F^*$, and finally maps that operate only within orbits. Thus, every proof of uniqueness can be restated as an algebraic proof with a certain $\Pi_0$ and an appropriate path. In the special case when $\Pi_0$ is a selection from Orb($F^*$), an algebraic proof requires finding a path within orbits only.

In the next section, we will apply Theorem 2 to obtain proofs of various well-known results and to assess the “efficiency” of any set of axioms. A set of axioms $\Omega$ is efficient when we cannot make any axiom from $\Omega$ weaker without affecting the uniqueness of $F^*$. If a set $\Omega'$ is weaker than $\Omega$ and if $\Omega'$ still implies the existence of a path implied by $\Omega$, then $\Omega$ is inefficient. It turns out that axioms from an independent set, such as May’s axioms, can be often significantly weakened. Efficiency is a more demanding requirement than independence. Staying within the set-theoretic interpretation of axioms, independence requires that every specific axiom in $\Omega$ must include at least one basic axiom that is not included in other axioms but it does not rule out overlap between axioms. Efficiency requires that all axioms in $\Omega$ must be pairwisely disjoint.

5. Uniqueness theorems: examples

In this section, we will analyze a few well-known uniqueness results from social choice and bargaining theory. The main purpose of every exercise is to find some stationary set $\Pi_0$ and a concise path to $\Pi_0$. Since criteria for finding an axiom “appealing” are rarely efficiency-motivated, it is no wonder that no axiomatic characterization discussed below is efficient. In all these cases, there are many paths for some problems, and one can construct many proofs based on different paths. Still, some paths look more appealing, and, regardless of the choice of a specific path, uncovering any path clarifies the structure of a proof.
A generic proof that $\Omega$ characterizes $F^*$, based on the results of this paper, should proceed as follows:

1. For some nonempty set $\Pi_0 \subset \Pi$, prove that for all $p \in \Pi_0$, $F(p) = F^*(p)$;
2. Find an appropriate family of maps $\Omega^*$ that is implied by $\Omega$, i.e., define $\Omega^*$ and then prove that every element of $\Omega^*$ is a map;
3. For a problem $p \in \Pi - \Pi_0$, find a path from $\Omega^*$ that sends $p$ into $\Pi_0$, i.e., prove that some sequence of maps from $\Omega^*$ transforms $p$ into an element of $\Pi_0$.

All proofs comprise similar steps, while the degree of difficulty of the main job proving that some function is a map varies. In all cases, maps are described parametrically, often with a permutation group.

Other components of informational analysis include calculating orbits and examining the efficiency of axioms. The results discussed below are well known, and we intend to only highlight the informational structure of proofs and axioms, not to obtain new results. It is assumed that the reader is familiar with these results. All proofs are outlined, and minor details are typically omitted.

5.1. May’s theorem and simple majority

May’s axiomatic characterization of simple majority rule provides our opening example. It opened a wide, and still underexplored, field of applications of social choice theory to normative political philosophy. Recall that $\Pi^M = \{(d_1, \ldots, d_n) : d_i \in \{-1, 0, 1\} \text{ for } i \in N = \{1, \ldots, n\}\}$ and $O^M = \{-1, 0, 1\}$. First, define simple majority rule formally. Denote $\sum_{i=1}^n d_i = \sum^p$.

$$F_{SM} = \begin{cases} 
1 & \text{when } \sum^p > 0 \\
0 & \text{when } \sum^p = 0 \\
-1 & \text{when } \sum^p < 0
\end{cases}$$

May’s Theorem says that $F_{SM}$ is a unique decisive rule that satisfies AN, NT, and PR (May, 1952). In the present framework, decisiveness is satisfied by all rules and $F_{SM}$ clearly satisfies the remaining three axioms. Thus, we need to prove the uniqueness of $F_{SM}$.

Since $F_{SM}$ is defined explicitly, it is simple to calculate $\text{Orb}(F_{SM})$.

$$\text{orb}(0, \ldots, 0) = P_0 = \{p \in \Pi : \sum^p = 0\};$$
$$\text{orb}(1, \ldots, 1) = P_+ = \{p \in \Pi : \sum^p > 0\};$$
$$\text{orb}(-1, \ldots, -1) = P_- = \{p \in \Pi : \sum^p < 0\}.$$

Now, let us apply Theorem 2. First, AN and NT imply that for all $p \in P_0$, $F(p) = F_{SM}(p)$. Next, the paths are established separately for the remaining two orbits of $F_{SM}$. PR transmits equality with $F_{SM}$ to $P_+$, and NT transmits it further to $P_-$.
**Proof.** Step 1: Let \( P_0 = P_0 \). For all \( p \in P_0 \), by AN, \( F(-p) = F(p) \); by NT, \( F(-p) = -F(p) \); thus, \( F(p) = F(-p) = 0 \).

Step 2: For any \( p \in P_+ \), define a PR map \( \omega_{-1}(p) = q \), where \( q \) is a problem \( p \) with 1 subtracted from \( p \)'s first coordinate greater than \(-1\). By PR, \( \omega_{-1} \) is a map since \( F(q) = F_{SM}(q) = 0 \), or \( F(q) = F_{SM}(q) = 1 \), implies \( F(p) = 1 = F_{SM}(p) \).

Neutrality map: \( \omega_{-pp} \) is a map for any \( p \in P_+ \), by NT.

Step 3: When \( \omega_{-1} \) is applied to \( p \in P_+ \), \( \sum_{-1}^p (p) \in \Pi_0 \).

Thus, a path for any \( p \in P_+ \) is \( \omega_{-1} \); a path for any \( p \in P_- \); \( \omega_{-1} \), \( \omega_{-pp} \). \( \Box \)

We can improve the efficiency of the axioms. For instance, AN is used only within \( P_0 \) for one specific permutation \( \sigma(p) = -p \). Thus, the parts of AN referring to other orbits or other permutations are superfluous.

May’s model and theorem are simple enough to make the informational structure of the underlying axioms clear regardless of whether the present framework is applied or not. In the next case, the benefits are more substantial.

5.2. Nash bargaining problem

Bargaining theory was the first axiomatic theory that was intended to provide characterizations of solutions that, from normative or descriptive points of view, seemed plausible. Twenty-five years after the seminal Nash (1950) contribution, the field was galvanized with a competing Kalai–Smorodinsky (1975) solution, and again, later, with a new characterization of Nash solution by Lensberg (1988). Our definition of an \( n \)-person bargaining model follows the Thomson and Lensberg’s (1989) setup.

Let \( N = \{1, \ldots, n\} \), where \( n \geq 2 \). Let \( \mathbb{R}_+^n = \{x \in \mathbb{R}^n: x_i > 0 \text{ for } i = 1, \ldots, n\} \) denote a strictly positive orthant of \( \mathbb{R}^n \) and let \( \mathbb{R}_+^n = \{x \in \mathbb{R}^n: x_i \geq 0 \text{ for } i = 1, \ldots, n\} \) denote a non-negative orthant of \( \mathbb{R}^n \). For \( x, y \in \mathbb{R}^n \), let \( x > y \) mean \( x_i > y_i \) for all \( i \in N = \{1, \ldots, n\} \); let \( x \geq y \) mean \( x_i \geq y_i \) for all \( i \in N \) and \( x_i > y_j \) for some \( j \in N \); and let \( x \geq y \) mean \( x_i \geq y_j \) for all \( i \in N \). For any set \( p \in \mathbb{R}_+^n \) and a permutation \( \sigma \in G^N \), \( \sigma(p) = \{y \in \mathbb{R}_+^n: y = (x_{\sigma(1)}, \ldots, x_{\sigma(n)}), (x_1, \ldots, x_n) \in p\} \). For any \( g \in \mathbb{R}_+^n \), \( x \in \mathbb{R}_+^n \), \( \lambda = (\lambda_1, \ldots, \lambda_n) \) and \( \lambda p = \{\lambda x: x \in p\} \).

\( P^\prime \) is a set of all problems \( p \) such that (i) \( p \) is a convex and compact subset of \( \mathbb{R}_+^n \); (ii) there exists \( x \in p \) such that \( x > 0 \); and (iii) for all \( x, y \in \mathbb{R}_+^n \), if \( x \in p \) and \( x \geq y \), then \( y \in p \).

\( O^\prime = \mathbb{R}_+^n \).

It is assumed that every rule satisfies \( F(p) \in p \). The Nash Bargaining Solution is defined as \( F_{NB}(p) = \text{ArgMax}_{x \in p} \sum_{i=1}^n x_i \). Convexity and compactness of \( p \) imply that \( F_{NB} \) is single-valued, i.e., \( F_{NB}(p) \in O^\prime \). The four familiar axioms are formulated below.

**Axiom 4.** Pareto Optimality (PO): for all \( p \in P^\prime \), for all \( x \in \mathbb{R}_+^n \), if \( x \geq F(p) \), then \( x \notin p \).

**Axiom 5.** Symmetry (SY): for all \( p \in P^\prime \), if \( p = \sigma(p) \) for all \( \sigma \in G^N \), then \( F(p) = F(p)_j \) for all \( i, j \in N \).

**Axiom 6.** Scale Invariance (SINV): for all \( p \in P^\prime \), for all \( \lambda \in \mathbb{R}_+^n \), \( F(\lambda p) = \lambda F(p) \).

**Axiom 7.** Nash’s Independence of Irrelevant Alternatives (NIIA): for all \( p, q \in P^\prime \), if \( p \subseteq q \) and \( F(q) \in p \), then \( F(p) = F(q) \).
Nash (1950) proved that \(F_{\text{NB}}\) is the unique rule that satisfies PO, SY, SINV, and NIIA for the class of all two-person bargaining problems (i.e., \(n=2\)). The fact that \(F_{\text{NB}}\) satisfies all Nash axioms is straightforward. Below, Theorem 2 is applied to sketch the proof that \(F_{\text{NB}}\) is the unique rule with such a property.

For all \(a, b > 0\), define \(I_{ab} = \{(x_1, x_2) \in \mathbb{R}^2_+ \mid bx_1 + ax_2 = ab\}\), and \(p_{ab} = \{(x_1, x_2) \in \mathbb{R}^2_+ \mid bx_1 + ax_2 \leq ab\}\). The segment \(I_{ab}\) connects points \((0, b)\) and \((a, 0)\) and includes all points from \(\mathbb{R}^2_+\) that satisfy \(bx_1 + ax_2 = ab\). The “triangle” problem \(p_{ab}\) includes all points from \(\mathbb{R}^2_+\) that are located on or below the segment \(I_{ab}\).

**Proof.** Step 1: Define \(\Pi_0 = \{p_{11}\}\). By PO and SY applied to \(p_{11}\), \(F(p_{11}) = (1/2, 1/2) = F_{\text{NB}}(p_{11})\).

Step 2: We will use two families of maps, parametrized in a convenient way.

Normalization map: \(o_{\text{SINV}}^\text{NB}(p) = (1/(2F_{\text{NB}}^1(p)), 1/(2F_{\text{NB}}^2(p)))p\); it is a map by SINV and by definition of \(F_{\text{NB}}\) and the set \(\Pi^2\), that together imply that \(F_{\text{NB}}(p) > 0\) for all \(p \in \Pi^2\) and \(i=1,2\). Map \(o_{\text{SINV}}^\text{NB}\) normalizes problem \(p\) with respect to \(F_{\text{NB}}\), i.e., it stretches or squeezes \(p\) in both dimensions in such a way that the Nash solution for the problem becomes \((1/2, 1/2)\).

Triangulation map: \(o_{\text{NIIA}}^\text{NB}(p) = (p_{2F_{\text{NB}}^1(p)}F_{\text{NB}}^1(p), p_{2F_{\text{NB}}^2(p)}F_{\text{NB}}^2(p))\); it is a map by a standard separation argument since the line \(2F_{\text{NB}}^1(p)x_1 + 2F_{\text{NB}}^2(p)x_2 = F_{\text{NB}}^1(p)F_{\text{NB}}^2(p)\) separates \(p\) and the hyperbola \(x_1x_2 = F_{\text{NB}}^1(p)F_{\text{NB}}^2(p)\) and \((F_{\text{NB}}^1(p), F_{\text{NB}}^2(p))\) is precisely the intersection of \(p\), \(p_{2F_{\text{NB}}^1(p)}F_{\text{NB}}^1(p)\), \(p_{2F_{\text{NB}}^2(p)}F_{\text{NB}}^2(p)\), and the hyperbola. Map \(o_{\text{NIIA}}^\text{NB}\) substitutes \(p\) with the unique triangle problem with the same Nash solution.

Step 3: It follows directly from the definitions of both maps that they substitute any \(p \in \Pi^2\) with the triangle problem whose solution is \((1/2, 1/2)\), i.e., with \(p_{11}\), regardless of which one is applied first. Thus for any \(p \in \Pi - \{p_{11}\}\), both \(o_{\text{SINV}}^\text{NB}, o_{\text{NIIA}}^\text{NB}\) and \(o_{\text{NIIA}}^\text{NB}, o_{\text{SINV}}^\text{NB}\) are paths (see Fig. 1).

The surprising simplicity of the path becomes less mysterious when one calculates \(\text{Orb}(F_{\text{NB}})\). The orbits can be parametrized by a selection \(\{p_{ab}\}\) and \(\text{orb}(p_{ab})\) includes precisely these subsets of \(p_{ab}\), that include \((F_{\text{NB}}^1(p_{ab}), F_{\text{NB}}^2(p_{ab}))\). Thus, \(\text{orb}(p_{ab})\) includes precisely those problems to which NIIA can be applied.

The highly regular Nash axioms directly imply the canonical axioms of Theorem 1. Every Nash axiom contributes to exactly one axiom, A1, A2, or A3, with \(\Pi_0 = \{p_{11}\}\) and the selection \(\{p_{ab}\} \_{a,b>0}\). This fact makes the proof simple. PO and SY are used only to assure that \(F(p_{11}) = (1/2, 1/2)\), which is exactly the content of A1. SINV is a variance axiom and defines isomorphism between \(\text{Orb}(F_{\text{NB}})\) and \(O \cap R^2_+\). Axiom A2 requires slightly less, i.e., that \(F\) remains equal to \(F_{\text{NB}}\) when the argument changes from \(p_{11}\) to \(p_{ab}\). Finally, an invariance axiom NIIA says that \(F\) remains constant within each orbit of \(F_{\text{NB}}\) when \(p\) is substituted with its subset \(q\). A3 weakens this requirement to all subsets of \(p_{ab}\) only. Thus, all Nash axioms can be substituted with their weaker counterparts.

The isomorphism implied by SINV is so natural that it is difficult to justify any other variance axiom as its substitute. Let us retain this axiom and explore what happens when the remaining axioms or the number of bargainers change. In all cases discussed below, the proof applies after slight modifications.

First, we can manipulate A1 and remove PO. Then, by SY, \(F(p_{11}) \in \{(x, x) : x \in [0, 1/2]\}\). On the other hand, NIIA and SINV imply that only points from the Pareto frontier \(l_{11}\) and \((0, 0)\) can be solutions to \(p_{11}\). Otherwise, we could squeeze the bargaining set slightly, so
that the solution would stay within the new set. By SINV, the solution would not move; by NIAA, the solution would be squeezed at the same ratio as the entire set. Thus, removing PO admits only (0, 0) and (1/2, 1/2) as solutions to $p_{11}$. Point (0, 0) can be removed with the following axiom due to Roth (1977).

**Axiom 8.** Strong Individual Rationality (STIR): for all $p \in \Pi$, $F(p) > 0$.

Substituting PO with STIR leads to Roth’s (1977) theorem: $F^{\text{NB}}$ is the unique bargaining solution that satisfies STIR, SY, NIIA, and SINV. Clearly, even a weaker axiom than STIR is sufficient: $F(p_{11}) \neq (0, 0)$.

Second, we can substitute SY and PO with STIR. Then, any point from the entire interior of $l_{11}$ can be defined as a solution to $p_{11}$, i.e., $F(p_{11}) = (\alpha, 1-\alpha)$, where $\alpha \in (0, 1)$. Thus, the family of axioms that, together with STIR, NIIA, and SINV characterize a unique rule is parametrized by $\alpha$: $F^\alpha(p_{11}) = (\alpha, 1-\alpha)$. Define $F^\alpha(p) = \text{ArgMax}_{x \in p} x_{1}^{\alpha} x_{2}^{1-\alpha}$. Then, for every parameter $\alpha$, $F^\alpha(p_{11}) = (\alpha, 1-\alpha)$ and $\omega_{\text{SINV}}^\alpha$, $\omega_{\text{NIIA}}^\alpha$ remains a path for any $p$ when $F^{\text{NB}}$ is
substituted in the respective definitions with $F^x$. Thus, STIR, NIIA, and SINV characterize the family of asymmetric Nash solutions to the bargaining problem, which is precisely the content of Harsanyi and Selten’s (1972) result, obtained in a multi-person setting.

Third, NIIA can be substituted with other axioms. The Kalai–Smorodinsky solution is defined as follows: for all $p \in \Pi$, let $\text{Max}(p) = \{\max\{x_1: x \in p\}, \ldots, \max\{x_n: x \in p\}\}$. $F^{KS}(p)$ is the maximal point in $p$ on the segment connecting $(0, \ldots, 0)$ and $\text{Max}(p)$. Kalai and Smorodinsky (1975) characterized KS for two-person bargaining problems with PO, SINV, SY, and the following axiom.

**Axiom 9.** Individual Monotonicity (IM): For all $p, q \in \Pi$, if $\text{Max}(p)_i = \text{Max}(q)_i$ for all $i \neq j$, and $\text{Max}(p)_j \geq \text{Max}(q)_j$, then $F_j(p) \leq F_j(q)$.

IM says that when a bargaining set expands in such a way that $\text{Max}(p)_i$ increases while $\text{Max}(p)_j$ remains constant for all other $i$, then $j$ is not hurt by the expansion.

$\text{Orb}(F^{KS})$ can be conveniently parametrized by rectangles $r_{ab}$ such that $a, b > 0$ and $r_{ab} = \{x: x \in (a, b)\}$. The proof of Kalai–Smorodinsky theorem is structurally very similar to the proof of Nash theorem discussed earlier. $\Pi_0$ includes the unit square $r_{11}$. In the first step, we establish by PO and SY that $F(r_{11}) = (1, 1) = F^{KS}(1, 1)$. For any $p \in \Pi$ a path from $p$ to $r_{11}$ can be defined as $\omega^{KS}_{\text{SINV}}, \omega^{KS}_{\text{IM}},$ where $\omega^{KS}_{\text{SINV}}(p) = 1/(F^1_{\text{KS}}(p)), 1/(F^2_{\text{KS}}(p)))p$ and $\omega^{KS}_{\text{IM}}(p) = r_{F^1_{\text{KS}}(p)F^2_{\text{KS}}(p)}$. The first map uses SINV and stretches or squeezes $p$ in both dimensions in such a way that the solution for the problem becomes $(1, 1)$. The second map converts $p$ into the rectangle that $p$ spans. Since IM implies that problems spanning identical rectangles have identical solutions, $r_{11}$ has the same solution as $p$.

Finally, all results stated above remain valid for $n \geq 2$ after minor changes in notation, since all two-person maps remain maps in multi-person settings. For any $n \geq 2$, Nash theorem becomes Harsanyi’s (1959) characterization of multi-person Nash bargaining solution.

### 5.3. Sen’s social choice

A rule based on individual utilities can be interpreted within the classical Arrowian framework if it remains invariant under all increasing transformations of individual utility. Such a requirement imposes strong informational constraints via an invariance axiom implicitly present in the Arrowian framework. Sen (1970) introduced a general framework that expanded the class of social choice problems considered within the Arrowian model. In Sen’s model, informational assumptions could vary so that the invariance axioms could represent many empirically meaningful or otherwise interesting relations. Among the consequences of this fine idea was a crop of papers that completed the “second revolution” in social choice, with contributions by Strasnick (1976); Hammond (1976); D’Aspremont and Gevers (1977); Deschamps and Gevers (1978); Maskin (1978); Roberts (1980a,b); and Blackorby et al. (1984). A few characterization results demonstrated that, within Sen’s framework, there exist rules that return an ordering of social alternatives while satisfying slightly modified Arrowian axioms. Below, we will analyze one of the best known results, due to D’Aspremont and Gevers (1977), and Milnor (1954).
Let $X=\{x_1,\ldots,x_m\}$ be a set of at least three alternatives and let $N=\{1,\ldots,n\}$ be a set of at least two individuals.

$\Pi_{N\times X}^N \subseteq \mathbb{R}^{N\times X}$ includes all utility profiles of individuals from $N$ over alternatives from $X$. Elements of $\Pi_{N\times X}^N$ will be also denoted $p=(u_1^p,\ldots,u_n^p)$, where $u_i^p: X \rightarrow \mathbb{R}$.

$O^X$ includes all social orderings, i.e., all reflexive, complete, and transitive binary relations over $X$.

A rule $R: \Pi_{N\times X}^N \rightarrow O^X$ is called a social welfare functional, SWFL. The ordering $R(p)$ is denoted as $R^p$, and its strict part is denoted as $R^p$. An example of an SWFL is the utilitarian rule, $F^U$, defined as follows:

$$F^U(p) = R^p \text{ such that } xR^py \iff \sum_{i=1}^n u_i^p(x) \geq \sum_{i=1}^n u_i^p(y).$$

Every Sen’s invariance axiom requires that $F$ is invariant under operation of a specific group. One such axiom is based on the group $G^{CU}=\{\sigma \in G^N: \sigma(u_1^\sigma,\ldots,u_n^\sigma)=(a_1+bu_1^1,\ldots,a_n+bu_n^n) \text{ for some } a_1,\ldots,a_n \text{ and } b>0\}$. $G^{CU}$ allows for linear transformations of utility functions with the slope positive and identical for all individuals. The corresponding axiom is formulated as follows.

**Axiom 10.** Cardinal Unit Comparability (CU): for all $p \in \Pi$, if $\sigma \in G^{CU}$, then $F(p) = F(\sigma(p))$.

This axiom is the cornerstone of d’Aspremont and Gevers’ characterization. It is complemented by modified classical Arrowian axioms translated into the new framework.

**Axiom 11.** Strong Pareto Principle ($P^*$): for all $p \in \Pi$, for any pair $x, y \in X$, if for all $i$: $u_i^p(x) \geq u_i^p(y)$, then $xR^py$. If, in addition for some $i$: $u_i^p(x) > u_i^p(y)$, then $xR^py$.

**Axiom 12.** Modified Arrow’s Independence of Irrelevant Alternatives (AIIA): For all $p,q \in \Pi$, for any pair $x, y \in X$, if $u_i^p(x)=u_i^q(x)$ and $u_i^p(y)=u_i^q(y)$ for $i=1,\ldots,n$, then $xR^py$ iff $xR^qy$.

**Axiom 13.** Anonymity (ANO): For all $p \in \Pi$, for all $\sigma \in G^N$, $F(p) = F(\sigma(p))$.

$P^*$ is a stronger version of Arrowian Weak Pareto Principle; AIIA is a modified version of the original Arrowian independence of irrelevant alternatives; and ANO is a stronger version of non-dictatorship. Unrestricted domain is assumed implicitly in our definition of SWFL.

The d’Aspremont and Gevers’ theorem says that: If $F$ satisfies $P^*$, AIIA, ANO, and CU, then $F=F^U$. Since $F^U$ clearly satisfies the four axioms, we can apply our three-step methodology. Step 2 is more complex than in the bargaining model and all inferences are only outlined. The source of complexity is the presence of two nonalgebraic axioms, $P^*$ and AIIA, that contribute to algebraic axioms in a nontrivial way.

Let us call a problem $q=(v_i^q) \in \Pi$ unanimous if, for all $x, y \in X$, it holds that either $v_i^q(x) \geq v_i^q(y)$ for all $i=1,\ldots,n$, or that $v_i^q(x) \leq v_i^q(y)$ for all $i=1,\ldots,n$.

**Proof.** Step 1: Define $\Pi_0=\{q: q \text{ is unanimous}\}$. $P^*$ implies that for all $q \in \Pi_0$, $F(q) = F^U(q)$.

Step 2: we will use two families of maps, parametrized by individuals, alternatives, and numbers. Any map from both families is defined for any problem.
**Pairwise Strong Anonymity (PSA) map** $\omega_j^i$: for all $x_k \in X$, for all $i,j \in N$, $\omega_j^i(p) = q$ such that $u^i_j(x_k) = u^j_i(x_k)$, $u^i_j(x_k) = u^j_i(x_k)$, and $u^i_j(x_k) = u^j_i(x_k)$ if $i \notin \{i,j\}$ or if $i \in \{i,j\}$ and $r \neq k$. Map $\omega_j^i$ applied to $p$ exchanges $u^i_j(x_k)$ with $u^j_i(x_k)$; it is a map by $P^*$, ANO and AIIA (see D’Aspremont and Gevers, 1977: 206, Lemma 4).

**Independence of Origin (IO) map** $\omega_{i,a}$: for all $i \in N$, for all $a \in \mathbb{R}$, $\omega_{i,a}(p) = q$ such that $u^i_j(x_k) = u^i_j(x_k) + a$ for all $x_k \in X$ and $u^i_j(x_k) = u^i_j(x_k)$ for all $j \neq i$ and for all $x_k \in X$. $\omega_{i,a}$ adds $a$ to $u^i_j$; it is a map by CU.

Step 3: For any $p \in \mathbb{P}$, our path consists of $4m(n-1)$ maps constructed from PSA and IO.

The superposition of four maps $\omega_{n,u^i_j(x_k)} \omega_{m,u_j^i(x_k)} \omega_{m,u^i_j(x_k)} \omega_{n,u_j^i(x_k)}$ is denoted by $\omega_j^i$. Map $\omega_j^i$ applied to $p$ substitutes $u^i_j(x_k)$ with $u^i_j(x_k) + u^i_j(x_k)$ and $u^i_j(x_k)$ with zero. Thus, $\omega_j^i$ transfers the utility of $j$ at $x^i_j$ to individual $n$.

Path: the sequence $\omega = \omega_1^1 \omega_2^1 \ldots \omega_{n-1}^1 \omega_m^1 \ldots \omega_m^n$, i.e., the sequence of all maps $\omega_j^i$ for $k=1,2,\ldots,m$ and $j=1,\ldots,n-1$. By construction, all utilities in $q = \omega(p)$ are equal to zero except for $u^i_j(x_k) = \sum_{j=1}^n u^i_j(x_k)$ for $k=1,\ldots,n$. Thus, $q$ is unanimous. $\square$

Comment: IO and a stronger version of PSA were introduced by Milnor (1954) as Column linearity and Symmetry, respectively, in the context of decision making under uncertainty. It is clear from the proof that the invariance axiom CU is too strong. The group $G^{CU}$ could be replaced with a smaller group $G^{IO} = \{\sigma \in G^{II} : \sigma(u_1^i, \ldots, u_n^i) = (a_1 + u_1^i, \ldots, a_n + u_n^i)\}$ for some $a_1,\ldots,a_n$. In addition, it is also clear that $F^U$ is the unique rule that satisfies $P^*$, IO, and PSA.

### 6. Conclusion

The present paper introduces algebraic axioms and investigates when a rule $F^*$ can be characterized by such axioms. The first result shows that every set of axioms that characterize $F^*$ is equivalent to a set of three algebraic axioms. The second result suggests a method for constructing an algebraic proof of uniqueness by finding an appropriate path of maps. It says that we can characterize a rule if and only if we can find a path. Both theorems remain valid for practically all characterization results in social choice and other social axiomatic theories. To illustrate their applications, they were used to prove and analyze various characterization results in May’s binary social choice, Nash’s bargaining theory, and Sen’s social choice theory.

The algebraic approach to axiomatic theories generates various specific benefits. First, it helps to prove a characterization theorem, to simplify an existing proof, and to see clearly the structure of a proof. Second, every parametrized family of maps represents an axiom. Thus, unless maps used in the proof are perfect translations of the “old” axioms, every maps-based proof generates a new characterization of a rule. Third, it can be easily seen which parts of an axiom are redundant in a particular characterization. Finally, connections between different characterizations can be uncovered. In some cases, various characterizations can be obtained in bundles, as was demonstrated for bargaining theory. Once the basic work is completed, the marginal effort required to uncover certain additional results may be surprisingly small.
Various questions remain open. If the removal of one axiom can convert an impossibility theorem into a uniqueness one, the present framework is obviously applicable. Note that this is not the case with Arrow’s theorem since if the Dictatorship is removed from the axioms, the remaining axioms characterize a family of rules, not a single rule (i.e., everybody can be a dictator and, in addition, there is some degree of freedom for the rule to decide what happens when the dictator is indifferent). Thus, a more general question is whether the present framework can generate useful insights into the nature of noncategoric theorems, i.e., when the axioms characterize a family of rules. Finally, another question remains open how useful is the framework for the characterizations of correspondences. A difficulty with correspondences seems to be that non-algebraic axioms are applied frequently and, for instance, may refer to the inclusion relation between outcomes.

The algebraic approach allows for a unified analysis of various models and the role played by axioms. Since every proof of uniqueness can be converted into an algebraic proof, it is not unthinkable that a sophisticated piece of software could be used to answer various characterization questions for some class of models.

References


